Market-Bound Research Contests

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Abstract

In many instances, the social value of an innovation is much larger than the profits that a firm can obtain by selling the innovation on the market. When this is the case, a research contest can help align incentives and increase welfare. This paper examines the optimal design of research contests when the objective of the contest designer is the discovery and broad adoption of socially valuable innovations. We show that the contest designer benefits from conditioning the size of the prize on the market performance of the winner. The optimal contest features two quantity cutoffs and two prize levels. The low prize is awarded if the winner sells a quantity greater than the first cutoff while the high prize is awarded if the winner sells a quantity greater than the second cutoff.

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1 Introduction

Much of innovative activity is driven by the desire of firms to improve their competitive position on the market. However, for many innovations the social value vastly surpasses any private value that could be captured on the market. This is the case for any innovation aimed at alleviating a negative externality. For example, overprescription of antibiotics could in the long term result in bacteria that are resistant to antibiotics. However, an individual patient would have a low willingness to pay for a test that would reduce the probability of such an outcome. Similarly, energy production creates pollution which is not internalized by the final consumers, hence they are less willing to pay for innovations which would make their automobiles and refrigerators more energy-efficient.

In cases like these, society can promote the beneficial innovation by holding an innovation contest. The examples above come from actual innovation contests – the 2015 Better Use of Antibiotics Prize, 2007 Automotive X Prize, and the 1992 Super-Efficient Refrigerator Prize (SERP) offered substantial monetary rewards in order to induce innovation in their targeted fields. One drawback of innovation contests is that the contestants will tailor their innovations to be appealing to the contest sponsors and not to the final consumers. This could end up hurting the contest sponsors as then the innovations produced by the contest remain unused. The organizers of the 2007 Automotive X Prize were aware of this issue, as the following statement from one of the Prize organizers reveals:

> "There are lots of competitions to make hyper-efficient cars – but often they look like rolling coffins. [...] We wanted a focus on consumer desirability."[1]

They tried to correct for this by instructing their expert jury to consider "consumer desirability" when judging contestants. However it is doubtful that a jury could evaluate accurately something like "consumer desirability." In fact, the contestants often have a better idea of what the consumers want than the contest sponsors do. Executives of Whirlpool, which won the 1992 SERP contest, were worried that the contest sponsors were misjudging the consumer sentiment: "[Contest sponsors] seemed to believe that consumers would flock to energy-efficient refrigerators even though Whirlpool's experience showed that wasn't true."[2]

An alternative would be to try to rely on some objective measure to judge contestants. Still, capturing the utility of an innovation with an objective measure is notoriously difficult. As a striking example, Netflix never used the winning algorithm of its $1 million contest because of the engineering costs – which were not part of the objective measure.[3]

[1]Cited in Murray, Stern, Campbell, and MacCormack (2012).

[2]See Treece J.B. (1993, July 5). The great refrigerator race. *Businessweek.* https://www.bloomberg.com/news/articles/1993-07-04/the-great-refrigerator-race.

[3]See Johnston, C. (2012, April 13). Netflix never used its $1 million algorithm due to engineering costs.

The sponsors of the 1992 SERP had a different idea – instead of trying to guess or measure product desirability, they gave the contestants an incentive to deliver products which will be desirable to consumers. Namely, in the 1992 SERP contest, the prize for the winner depended on the number of refrigerators sold. The contest winner, Whirlpool, received around \$120 from the contest sponsors for each refrigerator of the winning type sold, with a maximum of \$30 million in total (Gillingham, Newell, and Palmer, 2004). Given such a prize scheme, Whirlpool had an incentive to develop a refrigerator that would both meet the requirements of the sponsors (in order to win the contest) and be desirable to consumers (in order to maximize the size of the payoff).

Offering a per-unit-subsidy is one specific (and ad hoc) way to design such a contest. The objective of this paper is to systematically analyze settings as above (where a sponsor wants to incentivize innovation, but the payoff of the sponsor is only realized if the consumers adopt the innovation) and to characterize the optimal contest. To do so, we develop a model with two possible innovations – a *desirability innovation* and an *externality innovation*. Consumers are heterogeneous. One group of the consumers values only the desirability innovation while a second, smaller group values only the externality innovation. Firms choose how they direct their research efforts, either focusing on one of the two possible innovations, or – in an attempt to develop a product that will appeal to both groups – splitting their research efforts in both directions. A laissez-faire environment will in general be inefficient, because the firms will tend to focus on the desirability innovation, which is appealing to the larger consumer group.

However, there is a principal who wishes to incentivize discovery and broad distribution of the externality innovation and for that end commits to a research contest. The winner of the contest will be the firm that discovers the externality innovation, but the size of the prize may depend on the number of units that the winning firm sells on the market. The contest design is exactly the design of a function which maps the number of units sold into a prize that the principal pays. This is a flexible setting, which includes all contests where the firm which discovers the externality innovation gets a fixed prize, but it also includes a per-unit subsidy as in the SEMP contest and many other contest formats.

A standard fixed-prize contest is in general suboptimal. While it can provide incentives for firms to focus on the externality innovation, it cannot reliably provide incentives for firms to attempt to discover both innovations, which is necessary for broad adoption of products featuring the externality innovation. In addition, a fixed-prize contest is suboptimal because it does not incentivize the winner to expand the supply of the winning product. A fixed-prize contest is an example of a research contest where the prize does not depend on the market outcomes. Broadly, our results show that this class of research

Ars Technica. https://arstechnica.com/gadgets/2012/04/netflix-never-used-its-1-million-algorithm-due-to-engineering-costs.

contest is inefficient and that *market-bound research contests* can perform better.

We show that the optimal contest belongs to the class of *two-cutoff contests*. A two-cutoff contests feature two prize levels. If the winner sells at least the quantity specified by the first cutoff, a smaller prize is paid. If the winner manages to sell the quantity specified by the second cutoff, then the larger prize is awarded. The second cutoff provides incentives for the research aimed at discovery of both innovations, because the cutoff is set in such a way that only a product that appeals to both groups of consumers can sell in sufficient numbers. The first cutoff plays a different role. Even if a firm attempts to discover both innovations, it might by chance fail to do so. As our results show, if the first prize is below a certain threshold value, it can increase the expected payoff of pursuing both innovations without inducing the firm to over-specialize and produce a "rolling coffin." As our result shows, by optimally choosing the two cutoffs and the two prize levels, the principal achieves the optimum.

Our results have clear implications for the design of research contests. We demonstrate the benefits of market-bound research contests for innovations that features significant externalities and that can be quickly commercialized if discovered. Examples of such innovations include more fuel-efficient vehicles and appliances, tests for the need to use antibiotics, and improvements in the vaccine formulation, storage and transportation technology, to name a few. We furthermore show that the optimum can be achieved with a simple two-cutoff contest, which can be straightforwardly implemented and explained to the contestants. Their simplicity also makes two-cutoff contests credible, as a court of law can verify whether the quantities sold meet the threshold requirements, making it difficult for the principal to renege of the promise to pay the prize.

2 Related literature

The seminal paper analyzing static innovation contests is Che and Gale (2003). They show that a mechanism resembling a scoring auction is optimal and stress the optimality of limiting the number of participants. Dong, Fu, Serena, and Wu (2024) also show that including only the top two firms is optimal if firms are heterogeneous and the organizer can collect entry fees and re-allocate research resources, thus adjusting the degree of heterogeneity. Terwiesch and Xu (2008) argue that a larger number of contestants can be beneficial, because it results in a larger set of proposed solutions, while Koh (2017) shows that the optimal number of participants might depend on the how uncertain the project qualities are: two participants for low uncertainty and multiple participants for high uncertainty. Regarding the reward size, Erkal and Xiao (2021) show that the optimal prize might depend on how scarce the high-quality ideas are. Schöttner (2008) shows that when innovation process is stochastic, a fixed-prize tournament can outperform a scoring auction.

An important feature, which is prevalent in research contests and which has been a subject of studies, is the variety and diversity of approaches and ideas. Letina and Schmutzler (2019) study the effects of contest design on the variety of approaches to innovation. They find that the fixed-prize tournaments do not provide incentives for differentiation, unlike scoring auctions which can implement optimal diversity. Letina and Schmutzler (2019) also show that the bonus tournament is optimal in this setting. In a similar setting, but when the firms are initially technologically diverse, Protopappas and Rietzke (2023) show that fixed-prize tournaments achieves the first-best outcome, if firms are equally flexible. Rank-order contests avoids the distortions, which might arise under fixed-prize tournaments with heterogeneous firms' flexibility. Carnehl and Schneider (2024) show that when the direction of scientific questions is chosen sequentially by a stream of researches, ideas might be not studied deeply enough. This distortion can be corrected, if a principal could choose the initial two directions far apart enough, so that the future researchers would be incentivized to "bridge the gap" in the knowledge. In a setting where diverse technologies can all lead to the same result when successful, Block (2023) stresses the importance of rewards, tailored to the specific approaches. Lemus and Temnyalov (2024) study the information design in a setting, where contestants might choose different research directions.

Dynamic innovation contests are analyzed in Taylor (1995). His main result is that in a fixed-price tournament each contestant does research until he reaches a certain individual threshold. This is clearly inefficient, as individual contestants exert costly effort even when a good innovation is already available. Benkert and Letina (2020) show that this inefficiency can be resolved by using an interim-prize contest. Halac, Kartik, and Liu (2017) study innovation contests in bandit settings, where learning is crucial. They find that revealing the winners only after multiple successes can be optimal because it prevents the contestants from becoming too pessimistic too soon.[4] Chen, Chen, and Knyazev (2022) and Chen and Liu (2024) also study information disclosure in the dynamic contests but without fundamental uncertainty of the success possibility. Chen et al. (2022) show the optimality of an immediate public announcement of a successful submissions under endogenously chosen prize. Chen and Liu (2024) study a similar environment in the presence of a leader, who has to make a single breakthrough, and a chased, who needs two breakthroughs.

The emergence of online innovation platforms has enabled researchers to study innovation contests empirically. Using data on software contests, Boudreau, Lacetera, and Lakhani (2011) and Boudreau, Lakhani, and Menietti (2016) show that increasing the number of contestants in general benefits the principal, because it tends to increase the chances of a high-quality outcome. Using data on prediction contests, Lemus and Marshall (2021) show that information revelation is very important. Namely, revealing the

[4]See also Bimpikis, Ehsani, and Mostagir (2019).

current standing in the contest (with some noise) benefits the principal. Connecting empirical study with the contest dynamics, Lemus and Marshall (2022) study the benefits of contingent prizes in dynamic settings. Combining the prediction contests data with lab experiments, they show how competition organizers can benefit from using score- and time-contingent prize structures. In our model, we also show the importance of contingent prize structures, albeit from a theoretical view. Finally, using a novel method to calculate originality, Gross (2020) studies the effect of competition on creativity in online logo contests.

Finally, the goal of this paper is to study the optimal contest design when both the innovation and the product market outcomes are taken into account. Both elements are present in Che, Iossa, and Rey (2021), Galasso, Mitchell, and Virag (2018), and Chari, Golosov, and Tsyvinski (2012). Che, Iossa, and Rey (2021) study the use of prizes and contracts to both induce innovation and then implement the innovation. The main concern there is that innovators have private information regarding their implementation costs. Galasso, Mitchell, and Virag (2018) compare contests in which participants receive a prize without a patent, a patent without a prize, and both a prize and a patent, and show how awarding both a prize and a patent can be beneficial if research direction and market performance are taken into account. Chari, Golosov, and Tsyvinski (2012) suppose that the innovator receives signals about the quality of innovation, and can potentially manipulate the signals. In our paper, the main concern is that innovators will select research direction which is more likely to be positively evaluated by the principal and not the direction which will deliver a marketable good.

Also related are Kremer (2000a) and Kremer (2000b) who propose offering subsidies for the sales of vaccines in developing countries (an instrument known as Advanced Market Commitment). Such subsidies in effect increase the market value of an innovation. Kremer et al. (2022) study the AMC formally in the context of a firm, developing a new vaccine which would benefit a government, with the development sponsored by a third party. A prevalent holdup problem can lead to the inefficiently low quantity of vaccine. Optimal design of an AMC helps to solve this holdup problem. When the vaccine development is close to completion, the shape of AMC-incentives is crucial for efficient supply quantity. In that case, similarly to results in our paper, Kremer et al. (2022) find that an AMC that makes the payment contingent only on meeting the quantity requirement allows to achieve efficient production. Our paper complements Che, Iossa, and Rey (2021), Galasso, Mitchell, and Virag (2018), Chari, Golosov, and Tsyvinski (2012) and the literature on AMCs by examining how the right kind of research direction can be induced with the use of a market-bound innovation contest.

3 The Model

Firms. There are two ex-ante symmetric firms $i \in \{1, 2\}$ that can innovate. Two types of innovations can be discovered in this product market. In line with the main motivating examples, we will refer to one innovation type as *desirability innovation* and the other as *externality innovation*.

Each firm possesses some research capacity that it can direct towards search for the desirability innovation, towards search for the externality innovation, or it can divide the research effort and search for both innovation types. We will denote these three innovation strategies as DD, EE and DE, so that the innovative activity choice by the firms is $a \in \{DD, EE, DE\}$. Let the tuple $(\bar{d}_i, \bar{e}_i) \in \{0, 1\}^2$ capture the outcome of firm i's innovation strategy, so that $\bar{d}_i = 1$ implies that firm i has discovered the desirability innovation, $\bar{d}_i = 0$ implies that it has not, with analogous notation for the externality innovation.

Conditional on the firm's decision, nature determines whether the particular innovation is successful or not. We assume that if the firm chooses DD the outcome is

$$(\bar{d}_i, \bar{e}_i) = \begin{cases} (1, 0), & \text{with probability } p, \\ (0, 0), & \text{with probability } 1 - p. \end{cases}$$

Similarly, if the firm chooses EE, the outcome is

$$(\bar{d}_i, \bar{e}_i) = \begin{cases} (0, 1), & \text{with probability } p, \\ (0, 0), & \text{with probability } 1 - p. \end{cases}$$

Finally, if the firm decides to invest into both the externality and desirability innovations, so that $a = DE$, the outcome is

$$(\bar{d}_i, \bar{e}_i) = \begin{cases} (1, 1), & \text{with probability } q^2, \\ (1, 0), & \text{with probability } q(1 - q), \\ (0, 1), & \text{with probability } q(1 - q), \\ (0, 0), & \text{with probability } 1 - 2q + q^2. \end{cases}$$

We assume that $p > q$. This assumption reflects the fact that it is usually more complicated to achieve a success when pursuing multiple research directions than when the focus is on a single research objective. We normalize the cost of any research activity a to 0.

We assume that both innovations are patentable. If a single firm discovers the innovation, it receives the patent for sure. If both firms discover the innovation, they receive the patent with an equal probability. A firm which does not discover the innovation cannot receive the patent. The patents that a firm i holds describe that firm's type $t_i \in T = \{d, e, de, \varnothing\}$, where $t_i = d$ denotes a firm with a desirability patent, $t_i = e$ a firm with an externality patent, $t_i = de$ a firm with both patents and $t_i = \varnothing$ a firm with no patents.

After the patents have been allocated, they become common knowledge. Afterwards, the firms simultaneously choose a single per-unit price for their products. The marginal production costs are normalized to zero.

Consumers. There is a unit mass of potential buyers. Each consumer can only buy one unit from one of the firms, or buy nothing at all. The consumers are heterogeneous in two dimensions, the vertical and the horizontal. Vertically, consumers differ in θ, their willingness to pay for a unit of quality. Denote by Θ all the possible values of willingness to pay in the population of consumers. Horizontally, consumers differ in what they perceive as quality: one group of consumers only values the desirability innovation, while the other, mutually exclusive group, only values the externality innovation. Label the former group δ, and the latter ε. Denote the horizontal component of the consumer's type by $\eta \in \{\delta, \varepsilon\}$. Thus, the type of the consumer is characterized by $(\theta, \eta) \in \Theta \times \{\delta, \varepsilon\}$.

Suppose that a firm i charges some price ρ_i for its product. Then, the utility of a consumer with the type (θ, δ) from buying the product from this firm is

$$U(i|\theta, \delta) = \theta \times \mathbb{I}_{\{t_i = d \vee t_i = de\}} - \rho_i.$$

Similarly, the utility of a consumer with the type (θ, ε) from buying this product is

$$U(i|\theta, \varepsilon) = \theta \times \mathbb{I}_{\{t_i = e \vee t_i = de\}} - \rho_i.$$

We assume that the willingness to pay, θ, and the horizontal component, η, are independently distributed. Further, we assume that $\Theta = [0, 1], \theta \sim \mathcal{U}_{[0,1]}$, and that $\mathbb{P}\{\eta = \delta\} = m_d$. The latter condition simply means that the mass of the sub-population, who value the desirability feature, is m_d, with the rest of the population valuing the externality feature. We will assume that $m_d > 1/2$. That is, the desirability innovation is valued by the majority of the population of consumers.

Contest design. There is a principal who can organize a research contest by committing to transfer a monetary reward to the firm that acquires the externality patent. The principal has an exogenously fixed maximal budget $B > 0$.

In line with the motivating examples, we assume that the quantity sold by either firm is observable and contractible. This enables the principal to commit not only to award a fixed prize to the contest winner, but to condition the prize on the market performance of the winner. Formally, a *market-bound contest* is a contract in which the principal commits to reward a firm which holds the patent to the externality innovation according to a non-decreasing reward function

$$b : [0, 1] \to [0, B] .$$

That is, if firm i obtains the patent to the externality innovation and sells $Q \in [0, 1]$ units, then the principal commits to pay $b(Q)$ to firm i. This formulation allows a *fixed-prize innovation contest*, where a fixed prize is paid to the winner of the contest, but it also allows for richer structures with the possibility that the prize depends on the market performance of the winning firm.

For example, if $b(Q) = P$ for all $Q \in [0, 1]$, then we have a fixed prize contest with the prize P. An alternative contest would be a per-unit subsidy, where $b(Q) = sQ$ for some $s > 0$. That is, the principal commits to paying a subsidy s for each unit sold by the winning firm. Note that the reward function b need not be continuous. One example is a contest where the winner must sell at least some quantity $Z > 0$ in order to obtain a fixed prize P:

$$b(Q) = \begin{cases} 0, & \text{if } Q < Z, \\ P, & \text{if } Q \geq Z. \end{cases}$$

As these examples show, the space of market-bound contests is very rich, as each reward function b results in a different market-bound contest. As will be shown later, the ability of the principal to condition the prize value on the market performance of the winner will be useful.

Given a contest reward function b, we can write the payoffs of the firms as follows. If a firm holds a patent t and a sets a price $\rho \in [0, 1]$, then that firm's payoffs are

$$\begin{aligned}\pi(\rho|t, b) =& m_d(1-\rho)\rho\mathbb{I}_{\{t=d\}} + \\ & [(1-m_d)(1-\rho)\rho + b((1-m_d)(1-\rho))]\,\mathbb{I}_{\{t=e\}} + \\ & [(1-\rho)\rho + b(1-\rho)]\,\mathbb{I}_{\{t=de\}}.\end{aligned}$$

Let $\rho^*(t, b)$ be a price which maximizes $\pi(\rho|t, b)$. Taking into account the move by nature, denote with $\mathbb{P}(t_i, t_j|a_i, a_j)$ the probability that the subgame with patent allocations (t_i, t_j) is reached and let

$$\mathbb{P}(t_i|a_i, a_j) = \sum_{t_j \in\, T} \mathbb{P}(t_i, t_j|a_i, a_j).$$

Then, the expected payoff of firm i, when it chooses research strategy a_i when the opponent is choosing a_j is

$$\Pi_i(a_i, a_j) = \sum_{t_i \in T} \mathbb{P}(t_i | a_i, a_j) \pi_i(\rho^*(t_i, b | t_i, b).$$

If a_1^* and a_2^* are mutual best responses for Π_1 and Π_2, then $(a_1^*, \rho_1^*(.), a_2^*, \rho_2^*(.))$ constitute a subgame perfect Nash Equilibrium (SPNE). Throughout, we will focus on pure strategy SPNEs.

Timeline. We can summarize the timeline as follows.

Period 0:

- Nature randomly and privately determines the success of research activities and the allocation of patents.

Period 1:

- The principal commits to a contest reward function b.

Period 2:

- Firms simultaneously choose the innovation activity a_i and a_j.
- The actions taken by Nature and both firms become public knowledge.
- Patents are allocated and firm types t_i and t_j are realized.

Period 3:

- Firms simultaneously choose prices ρ_i and ρ_j.
- Payoffs are realized.

4 Optimal contest

The problem of fining the optimal contest in this setting is in general challenging, because it would involve optimizing the designers objective function over the set of all admissible functions b. However, as our results in the Section 4.1 show, we can, with minimum loss of generality, focus on a class of much simpler contests, which we call two-cutoff contests. In the following subsection we then apply this result to find the optimal contest when the designers objective is to minimize the budget needed to induce the firms to pursue both innovations.

4.1 Optimality of two-cutoff contests

A two-cutoff contest is defined by: (i) two quantity cutoffs Q^I and Q^{II} where $0 \leq Q^I \leq Q^{II} \leq 1$, (ii) two prize levels Z^I and Z^{II} where $0 \leq Z^I \leq Z^{II} \leq B$, and (iii) a reward function b^{tc}, where

$$b^{tc}(Q) = \begin{cases} 0, & \text{if } Q \in [0, Q^I), \\ Z^I, & \text{if } Q \in [Q^I, Q^{II}), \\ Z^{II}, & \text{if } Q \geqslant Q^{II}. \end{cases}$$

That is, in a two-cutoff contest there are two potential prizes: a smaller prize Z^I and a larger prize Z^{II}. To qualify for the smaller prize, the winner of the contest has to sell at least Q^I units. To qualify for the larger prize, the winner has to sell at least Q^{II} units.

Optimizing over the set of two-cutoff contests is significantly easier than optimizing over arbitrary market-bound contests. Instead of having to specify an entire function b, the problem of finding the optimal two-cutoff contest boils down to the choice of four real-valued variables: Q^I, Q^{II}, Z^I, and Z^{II}.

Moreover, as our next result shows, limiting the attention to two-cutoff contests is – for essentially all plausible objectives of the principal – without loss.

Proposition 1 (Optimality of two-cutoff contests)**.**
Fix any contest, any SPNE of that contest $(a_1^*, \rho_1^*(), a_2^*, \rho_2^*())$, and any pricing function $\hat{\rho}(t)$ such that $\hat{\rho}(t) \in \{\rho_1^*(t), \rho_2^*(t)\}$ for all $t \in T$. Then, there exists a two-cutoff contest with an SPNE $(a_1^{tc}, \rho_1^{tc}(), a_2^{tc}, \rho_2^{tc}())$ such that:

(i) $a_1^{tc} = a_1^*$ and $a_2^{tc} = a_2^*$;

(ii) $\rho_1^{tc}(t) = \rho_2^{tc}(t) = \hat{\rho}(t)$ for all $t \in T$.

In the appendix, we provide a constructive proof. To understand the proposition, first note that the only restriction on what can be implemented with a two-cutoff contest (relative to what can be implemented with an arbitrary contest) is that the pricing behavior of firms has to be symmetric, i.e. $\rho_1^{tc}() = \rho_2^{tc}()$. If it happens that in the arbitrary contest the pricing behavior is already symmetric, i.e., that $\rho_1(t) = \rho_2(t)$ for all $t = T$, then an immediate corollary of Proposition 1 is that such an equilibrium can be implemented in a two-cutoff contest. As the proof of the proposition shows, the pricing behavior of firms will be symmetric whenever the contest reward function b is such that $\pi(\rho|t, b)$ has a unique maximizer.

If, for some t, the equilibrium pricing behavior is not symmetric, so that $\rho_1(t) \neq \rho_2(t)$, the argument above implies that both firms are indifferent between $\rho_1(t)$ and $\rho_2(t)$, so that the principal can design a two-cutoff contest that implements *either* $\rho_1(t)$ *or* $\rho_2(t)$. Thus, as long as the objective of the principal does not depend on the asymmetry of pricing behavior of firms (which will be the case whenever the principal's payoff only depends on the realized innovation and quantities sold), then the set of optimal contests for the principal will always include a two-cutoff contest. In other words, in most reasonable situations it will be without a loss to focus only on two-cutoff contests. In particular, this will be the case for both objectives that we will examine below.

To understand why the proposition is true, suppose that we start with some arbitrary contest reward function b. Note that there can be two "types" of firms which can win the contest – either the firm with the patent $t = e$ or the firm with the patent $t = de$. Once a firm wins a contest, it then optimally chooses the price which maximizes either $\pi(\rho|e, b)$ or $\pi(\rho|de, b)$. Call these prices $\rho^*(e)$ and $\rho^*(de)$ and let the quantities that the firm sells at those prices be Q^I and Q^{II}. Now consider a two-cutoff contest where the quantity cutoffs are exactly Q^I and Q^{II} and rewards for reaching the cutoffs are the same as in the initial contests, that is, $b(Q^I)$ and $b(Q^{II})$. A firm which wins this two-cutoff contest then optimally chooses to set prices equal to $\rho^*(e)$ and $\rho^*(de)$, depending on which patent it holds. There can not be a profitable deviation from these prices, because if there were, it would also be a profitable deviation in the initial contest with the reward function b, which is not the case. Thus, with the caveat on asymmetric pricing strategies discussed previously, anything that can be implemented with an arbitrary contest can also be implemented with a two-cutoff contest.

A two-cutoff contest suffices (and not three- or four-cutoff contest) exactly because there are only two firm types which can win the contest. If there were more possible winner types, then more cutoffs would be needed for the result to hold. At the same time, this argument suggests that this result would hold in a model with more than two firms, with a continuous choice of research direction, and with more than two groups of consumers.

Moreover, generally there will be many contests that implement equivalent outcomes. The reason for this is that, starting from any two-cutoff contest, a small change in the reward function b^{tc}, at points which are sufficiently away from the quantities Q^I and Q^{II}, will not result in a change of pricing behavior by the firms, because a marginal change in the reward from the contest is not sufficient to compensate the firm for the lost market profits caused by a change in pricing behavior.

4.2 Direction of innovation

Consider now a contest designer that is interested in "fixing" the direction of innovation that firms choose. That is she would like the firms to pursue an innovation strategy that can result in a product that solves the externality problem without being a "rolling coffin." In other words, the contest designer is interested in implementing an equilibrium where both firms choose DE strategy in the first stage. This section has two goals. The first is to characterize the minimum budget B that the contest designer needs in order to implement (DE, DE) as part of an equilibrium. The second is to examine whether contests which *do not* condition on the market performance of the winner can be used as a tool to incentivize firms to choose DE as their research strategy.

The nature of the innovation production technology turns out to play a key role. We will differentiate between two cases. First, the case $p > 2q$, , which we refer to as *increasing returns to specialization.* In this case, a firm which does not specialize and chooses strategy DE will discover either innovation with probability q. If this firm specialized instead (for example, by choosing strategy DD), this doubling of its efforts towards the innovation $\bar{d}$ would lead to a more than double increase in the probability of discovering $\bar{d}$, from q to p. The other case, $p \leq 2q$ features *decreasing returns to specialization.* In this case, doubling of efforts by specializing in one direction results in a less than two times higher probability of discovery. As our next result shows, these two cases require different designs of innovation contests.

Proposition 2 (Minimal budget)**.**

(i) If $p > 2q$*,* (DE, DE) *is implementable iff* $B \geq \frac{m_d(p-q(2-2q+q^2))}{2(2-q)q^2} - \frac{1}{4}$*. A contest implementing it is* $Q^I = \frac{1-m_d}{2}$*,* $Q^{II} = \frac{1}{2}$*,* $Z^I = \frac{2m_d-1}{4}$ *and* $Z^{II} = \frac{m_d(p-q(2-2q+q^2))}{2(2-q)q^2} - \frac{1}{4}$*.*

(ii) If $p \leq 2q$*,* (DE, DE) *is implementable iff* $B \geq \frac{m_d p - q}{4q}$*. A contest implementing it is* $Q^I \leq Q^{II} \leq \frac{1-m_d}{2}$ *and* $Z^I \leq Z^{II} = \max\left\{\frac{m_d p - q}{4q}, 0\right\}$*.*

When $p > 2q$ firms have an incentive to specialize, as specialization increases their chances of discovering an innovation.[5] To overcome this incentive, the contest designer has to offer the firms a reward which can only be won if the firm in fact chose DE research strategy. This can be done by conditioning the reward on the firm selling a quantity that is only consistent with the firm holding the *de* patent and thus having a product which appeals to the entire market. Optimally, this is $Q^{II} = 1/2$, the monopoly quantity of the firm holding the *de* patent. The proposition then specifies Z^{II}, the lowest prize (and hence the minimum budget required) that incentivizes the firms to choose (DE, DE) – and thus not to specialize – in equilibrium.

The above intuition does not explain why Z^I, which represents the prize that the firm with *e* patent wins, is positive. To understand this, suppose that $Z^I = 0$ and that Z^{II} was such that (DE, DE) was an equilibrium, but that it would not be an equilibrium after a marginal reduction in Z^{II}. Lowering Z^{II} makes deviations to both DD and EE more desirable, but since $m_d > 1/2$ (the market for the good with the d patent is strictly larger), deviations to DD are strictly more profitable than deviations to EE. Thus, a marginal reduction in Z^{II} makes only the deviation to DD profitable, while the deviation

[5]As our results in the Online Appendix B.1 show, firms choose either DD or EE in any equilibrium when there is no contest and $p > 2q$.

to EE remains unprofitable. An increase in Z^I (which is awarded to the firms with the e patent) makes deviations to DD less profitable, while making the deviations to EE more profitable. Thus, starting from $Z^{II} > 0$ and $Z^I = 0$ it is always possible to slightly increase Z^I and decrease Z^{II} while keeping (DE, DE) as an equilibrium. Such a contest requires a strictly lower budget to be implemented, since $Z^I \leq Z^{II} \leq B$. This logic holds until $Z^I = \frac{2m_d - 1}{4}$, which is the optimal Z^I identified in the proposition.

When $p \leq 2q$, inducing (DE, DE) is much more straightforward. If $m_d \leq q/p$, the sizes of the two markets are sufficiently similar and since specialization is inefficient, (DE, DE) is an equilibrium without any intervention by the designer. In this case, optimal prize is $Z^I = Z^{II} = 0$. When $m_d > q/p$, the only reason why (DE, DE) is not an equilibrium is that the rewards from holding the patent e are not high enough. This can be solved by simply increasing the rewards to holding the patent e, without any need to condition on the market performance of the winning firm. Indeed, Proposition 2(ii) shows that not only can a standard *fixed-prize contest* be used to implement the (DE, DE), but also that an optimal fixed-prize contest does so with the smallest required budget. Thus, when $p \leq 2q$, so that the innovation technology features decreasing returns to specialization, then the "rolling coffins" problem mentioned in the introduction does not arise. This is not the case when $p > 2q$ (and the innovation technology features increasing returns to scale), as our next result shows.

Proposition 3. *Suppose that $p > 2q$. Then, given any fixed-prize contest, (DE, DE) is not an equilibrium.*

We already know from Proposition 2(i) that when the innovation technology features increasing returns to specialization, the contest which implements (DE, DE) with a minimal budget is a two-cutoff contest with two distinct prizes. Proposition 3 shows that this result is even stronger: with increasing returns to scale, a fixed-prize contest cannot implement (DE, DE), not only with the minimal budget but *with any budget at all.* Thus, solving the "rolling coffins" problem requires conditioning of the contest prize on the market performance of the winner – in other words, it requires that the contest designer uses a market-bound research contest.

The principal economic insight from this section is that there is a fundamental difference between innovation technologies with increasing and decreasing returns to specialization – and that the optimal contest design will change depending on the technological environment in which the desired innovation is supposed to be developed. When innovation technology features decreasing returns to specialization, the problem is easier. To incentivize the "right" direction of innovation, the designer needs to only increase the rewards to the discovery of the desired innovation. This can be done with a simple fixed-prize contest. However, when the innovation technology features increasing returns to specialization, then relying on fixed-prize contests is not enough. At most, a fixed-prize contest will incentivize firms to pursue the development of "rolling coffins" as was the case in the example from the introduction. To optimally direct the innovative efforts of firms, the designer has to condition the size of the prize on the market performance of the winner. This can be done with a simple two-cutoff contest. Moreover, if designed optimally, such a contest implements the desired direction of innovation with the lowest possible budget.

5 Conclusion

Research contests are a valuable policy tool, especially for incentivizing research into innovations that reduce negative externalities (or increase positive ones). Many innovations share these features and indeed several past research contests, like the 2015 Better Use of Antibiotics Prize, 2007 Automotive X Prize, and the 1992 Super-Efficient Refrigerator Prize, are aimed at exactly such innovations. Standard research contests, where a fixed prize is paid to a firm that successfully innovates (according to pre-specified criteria) induce firms to search for an innovation that would appeal to the contest sponsor, but not necessarily one that would be appealing to the broader consumer base. These are the "rolling coffins" mentioned in the introduction.

We propose that contest organizers instead use market-bound research contests, where the size of prize depends on the market performance of the winner. We show that a simple contests – what we call a two-cutoff contest – is optimal. The presence of two prize levels and two quantity cutoffs optimally aligns the incentives of the firms and the contest designer.

One advantage of fixed-prize contests, and partly a reason for their popularity, is that they are simple and transparent. A two-cutoff contest is not significantly more complex, neither from the perspective of contest design not from the perspective of explaining the rules to the contestants. Furthermore, it seems plausible that a court of law would be able to enforce such a contest. Given the growing popularity of research contests, it is important that sponsors and designers implement the designs which are optimal for the problem at hand. We believe that market-bound contests generally, and two-cutoff contests in particular, are a valuable addition to the set of tools that can be used to induce research which is socially valuable but underappreciated by the marketplace.

A Proofs

A.1 Proof of Proposition 1

Take any contest reward function b and any tuple $(a_1, \rho_1, a_2, \rho_2)$ such that $(a_1, \rho_1, a_2, \rho_2)$ constitute an SPNE given $b()$, where (a_1, a_2) are first period actions and (ρ_1, ρ_2) are the functions determining the second period prices.

Since ρ_1 and ρ_2 maximize $\pi(\rho_i|t_i, b)$, it must be that $\pi(\rho_1(t)|t, b) = \pi(\rho_2(t)|t, b)$ for all $t \in T$.

Let Q^I and Q^{II} be quantities such that $Q^I \in \{(1-\rho_1(e))(1-m_d), (1-\rho_2(e))(1-m_d)\}$ and $Q^{II} \in \{(1-\rho_1(de)), (1-\rho_2(de))\}$.

Lemma 1. $Q^I \leqslant Q^{II}$

Proof. Suppose not. Then $Q^I > Q^{II}$, which implies:

$$\max\{(1-\rho_1(e))(1-m_d), (1-\rho_2(e))(1-m_d)\} > \min\{(1-\rho_1(de)), (1-\rho_2(de))\}$$
$$\min\{\rho_1(e), \rho_2(e)\} =: \rho_m(e) < \rho_m(de) := \max\{\rho_1(de), \rho_2(de)\}.$$

Denote with ρ' the solution to $(1-m_d)(1-\rho') = 1-\rho_m(de)$. Note that a solution to this equality always exists because by assumption $Q^I > Q^{II}$. Since $\rho_m(e)$ is part of an equilibrium and thus maximizes payoffs, the following must be true

$$(1-m_d)(1-\rho_m(e))\rho_m(e) + b((1-m_d)(1-\rho_m(e)) \geqslant (1-\rho_m(de))\rho' + b(1-\rho_m(de)). \tag{$\star$}$$

Since $(1-m_d)(1-\rho') = 1-\rho_m(de)$, it is straightforward that

$$\rho' < \rho_m(de). \tag{$\star\star$}$$

Denote with ρ'' the solution to $(1-\rho'') = (1-m_d)(1-\rho_m(e))$. Again, since $\rho_m(de)$ is part of an equilibrium and thus maximizes payoffs

$$(1-\rho_m(de))\rho_m(de) + b(1-\rho_m(de)) \geqslant (1-m_d)(1-\rho_m(e))p'' + b((1-m_d)(1-\rho_m(e))). \tag{$\star\star\star$}$$

Clearly, $\rho'' > \rho_m(e)$, so that $(\star\star\star)$ and $(\star)$ imply

$$(1-\rho_m(de))\rho_m(de) + b(1-\rho_m(de)) > (1-\rho_m(de))\rho' + b(1-\rho_m(de))$$
$$\Longleftrightarrow$$
$$\rho_m(de) > \rho',$$

which is a contradiction to ($\star\star$). $\square$

Next, let

$$b^{TC}(Q) = \begin{cases} 0, & \text{if } Q < Q^I \\ b(Q^I), & \text{if } Q^I \leqslant Q < Q^{II} \\ b(Q^{II}), & \text{else.} \end{cases}$$

Consider strategies $(a_1^{TC}, \rho_1^{TC}, a_2^{TC}, \rho_2^{TC})$, where $a_i^{TC} = a_i$ and $\rho_1^{TC}(t) = \rho_2^{TC}(t) = \hat{\rho}(t)$ for all $t \in T$, where $\hat{\rho}(t) \in \{\rho_1(t), \rho_2(t)\}$ for all $t \in T$. We will show that $(a_1^{TC}, \rho_1^{TC}, a_2^{TC}, \rho_2^{TC})$ constitute an SPNE given b^{TC}, which completes the proof of the proposition.

First, since $\pi(\rho_1(t)|t, b) = \pi(\rho_2(t)|t, b)$, then, by construction,
$\pi(\rho_i(t)|t, b) = \pi(\rho^{TC}(t)|t, b^{TC})$ for all t and i.

Second, since $b^{TC}(Q^I) = b(Q^I)$, $b^{TC}(Q^{II}) = b(Q^{II})$ and $b()$ is non-decreasing, then

$$b^{TC}(Q) \leqslant b(Q) \quad \forall q \in [0, 1].$$

This in turn implies that for any ρ and t,

$$\pi(\rho|t, b) \geqslant \pi(\rho|t, b^{TC}).$$

Thus, if $\pi(\rho_i(t)|t, b)$ are the maximal payoffs under contest b, then ρ^{TC} maximizes payoffs under contest b^{TC} in every subgame in the second period. Since payoffs in every subgame in the second period are unchanged, then $a_1^{TC} = a_1$ and $a_2^{TC} = a_2$ maximize expected payoffs in the first period. Thus, $(a_1^{TC}, \rho_1^{TC}, a_2^{TC}, \rho_2^{TC})$ constitute a SPNE under b^{TC}.

A.2 Proof of Proposition 2

By Proposition 1, if a contest implements (DE, DE), then there exists a two-cutoff contests which also implements (DE, DE) and which does not require a higher budget to do so. Thus, without loss of generality, we can focus on implementation with two-cutoff contests.

Fix a two-cutoff contest $b^t c$ with cutoffs and prizes $(Q^I, Q^{II}, Z^I, Z^{II})$ and suppose for the moment that if a firm holds the patent e it indeed produces quantity Q^I while if it holds the patent de it produces quantity Q^{II}. Also suppose that firm j chooses DE. Then, the payoff of firm i from choosing research directions $a \in \{DE, DD, EE\}$ is given by:

$$\begin{aligned}
V(DE,DE) &= \mathbb{P}(d|DE,DE)\frac{m_d}{4} + \mathbb{P}(e|DE,DE)\left(Q^I\left(1-\frac{Q^I}{1-m_d}\right)+Z^I\right)+\\
&\quad + \mathbb{P}(de|DE,DE)\left(Q^{II}\left(1-Q^{II}\right)+Z^{II}\right),\\
V(DD,DE) &= \mathbb{P}(d|DD,DE)\frac{m_d}{4},\\
V(EE,DE) &= \mathbb{P}(e|EE,DE)\left(Q^I\left(1-\frac{Q^I}{1-m_d}\right)+Z^I\right).
\end{aligned}$$

The probabilities of different patent allocations $t \in \{d,e,de\}$, given the choice of research direction (a_i,a_j) are given by:

$$\begin{aligned}
\mathbb{P}(d|DE,DE) &= q(1-q)\left(q(1-q)+1-2q+q^2+\frac{q}{2}\right)+q^2\left(\frac{q(1-q)}{2}+\frac{q^2}{4}\right)\\
&= \frac{1}{4}(2-q)q\left(2-(2-q)q\right),\\
\mathbb{P}(e|DE,DE) &= \mathbb{P}(d|DE,DE),\\
\mathbb{P}(de|DE,DE) &= q^2\left(1-2q+q^2+\frac{q(1-q)}{2}+\frac{q(1-q)}{2}+\frac{q^2}{4}\right)\\
&= \frac{q^2(2-q)^2}{4},\\
\mathbb{P}(d|DD,DE) &= p\left(q(1-q)+1-2q+q^2+\frac{q}{2}\right)\\
&= p\left(1-\frac{q}{2}\right),\\
\mathbb{P}(e|EE,DE) &= \mathbb{P}(d|DD,DE).
\end{aligned}$$

Still assuming that a firm which holds the patent e produces quantity Q^I while if it holds the patent de it produces quantity Q^{II}, we can write the problem of finding the minimal budget needed to implement (DE,DE) as:

$$\begin{aligned}
\min_{Q^I,Q^{II},Z^I,Z^{II}} \quad & Z^{II}\\
\text{s.t. } V(DE,DE) &\geqslant V(DD,DE)\\
V(DE,DE) &\geqslant V(EE,DE)\\
Q^{II} &\geqslant Q^I\\
Z^{II} &\geqslant Z^I \geqslant 0.
\end{aligned}$$

Note that among all payoffs V, Q^{II} and Z^{II} are only present in $V(DE,DE)$, and that only through the expression $\left(Q^{II}\left(1-Q^{II}\right)+Z^{II}\right)$. Thus minimizing Z^{II} implies maximizing $Q^{II}\left(1-Q^{II}\right)$, or in other words setting Q^{II} equal to the monopoly quantity for the firm with the de patent. Further, Q^I and Z^I only enter the constraints through the expression

$\left(Q^I\left(1-\frac{Q^I}{1-m_d}\right)+Z^I\right)$, and the lowest payoff the firm with the e patent obtains is its monopoly payoff. Hence, the largest range of values that $\left(Q^I\left(1-\frac{Q^I}{1-m_d}\right)+Z^I\right)$ can take is given by setting Q^I to the monopoly quantity and choosing $Z^I\in[0,Z^{II}]$. It is also clear that if Q^I and Q^{II} are set to monopoly quantities, then the firms will also have an incentive to choose those quantities when they hold the patents e and de.

Setting Q^I and Q^{II} to monopoly quantities, we can rewrite the minimization problem as:

$$\begin{aligned}
\min_{Z^I,Z^{II}} \quad & Z^{II}\\
\text{s.t. } & V(DE,DE)\geqslant V(DD,DE)\\
& V(DE,DE)\geqslant V(EE,DE)\\
& Z^{II}\geqslant Z^I\geqslant 0.
\end{aligned}$$

Substituting the monopoly profits ($(1-m_d)/4$ with patent e and $1/4$ with patent de) and the expressions for V from above we obtain:

$$\begin{aligned}
&\min_{Z^I,Z^{II}} \quad Z^{II}\\
&\text{s.t.}\\
&\frac{1}{4}(2-q)q\left(2-(2-q)q\right)\left(\frac{m_d}{4}+\frac{1-m_d}{4}+Z^I\right)+\frac{q^2(2-q)^2}{4}\left(\frac{1}{4}+Z^{II}\right)\\
&\qquad\geqslant p\left(1-\frac{q}{2}\right)\frac{m_d}{4}\\
&\frac{1}{4}(2-q)q\left(2-(2-q)q\right)\left(\frac{m_d}{4}+\frac{1-m_d}{4}+Z^I\right)+\frac{q^2(2-q)^2}{4}\left(\frac{1}{4}+Z^{II}\right)\\
&\qquad\geqslant p\left(1-\frac{q}{2}\right)\left(\frac{1-m_d}{4}+Z^I\right)\\
&Z^{II}\geqslant Z^I\geqslant 0.
\end{aligned}$$

This is a linear minimization problem.

For $p<2q$, the minimization set can be restated as

$$Z^{II}\geqslant\begin{cases}\frac{m_dp-2q(2-2q+q^2)Z^I-q}{2(2-q)q^2}, & \text{if } Z^I<\frac{m_dp-q}{4q},\\ Z^I, & \text{if } Z^I\in\left[\frac{m_dp-q}{4q},\frac{q-(1-m_d)p}{4(p-q)}\right),\\ \frac{(1-m_d)p+2(2p-2q+2q^2-q^3)Z^I-q}{2(2-q)q^2}, & \text{if } Z^I\geqslant\frac{q-(1-m_d)p}{4(p-q)},\end{cases}$$

and $Z^I\geq 0$.

Then, the minimum is achieved at $Z^I=\max\left\{\frac{m_dp-q}{4q},0\right\}$, and the minimum required budget is the same: $Z^{II*}=\max\left\{\frac{m_dp-q}{4q},0\right\}$.

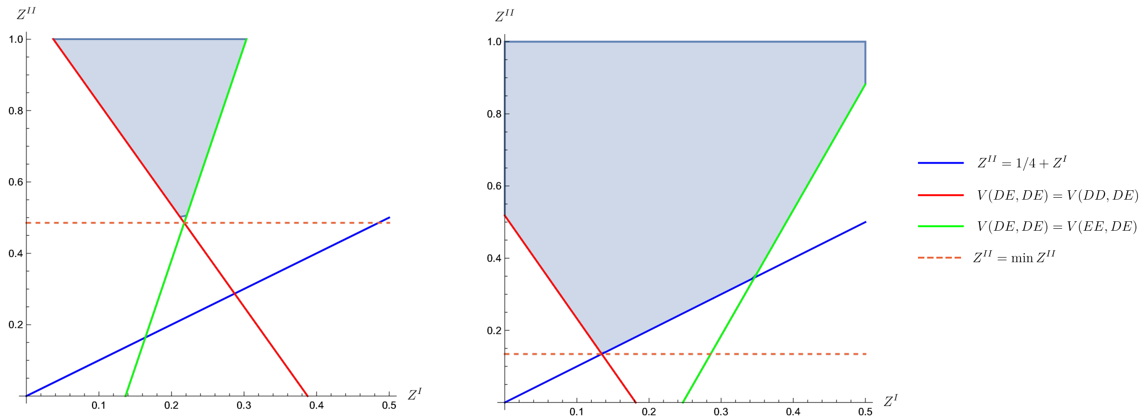
For $p \geqslant 2q$, the restated minimization set can be written as

$$Z^{II} \geqslant \begin{cases} \frac{m_d p - 2q(2-2q+q^2)Z^I - q}{2(2-q)q^2}, & \text{if } Z^I < \frac{2m_d-1}{4}, \\ \frac{(1-m_d)p + 2(2p-2q+2q^2-q^3)Z^I - q}{2(2-q)q^2}, & \text{if } Z^I \geqslant \frac{2m_d-1}{4}. \end{cases}$$

The minimum is then achieved at $Z^I = \frac{2m_d-1}{4}$, and is equal to $Z^{II*} = \frac{m_d(p-q(2-(2-q)q))}{2(2-q)q^2} - \frac{1}{4}$. Figure (1) illustrates the argument.

Figure 1: Budget minimization region and the minimum

(a) $p \geq 2q$ (b) $p < 2q$



A.3 Proof of Proposition 3

Notice that any fixed-prize contest can be represented as a two-cutoff contest with $Z^I = Z^{II} = Z^*$. Consider the feasible set of prizes from the proof of Proposition 2. If some fixed-prize contest could induce (DE, DE) as an equilibrium under $p > 2q$, it would hold that

$$\begin{aligned} &\frac{1}{4}(2-q)q\left(2-(2-q)q\right)\left(\frac{m_d}{4} + \frac{1-m_d}{4} + Z^*\right) + \frac{q^2(2-q)^2}{4}\left(\frac{1}{4} + Z^*\right) \\ &\qquad \geqslant p\left(1 - \frac{q}{2}\right)\frac{m_d}{4}, \\ &\frac{1}{4}(2-q)q\left(2-(2-q)q\right)\left(\frac{m_d}{4} + \frac{1-m_d}{4} + Z^*\right) + \frac{q^2(2-q)^2}{4}\left(\frac{1}{4} + Z^*\right) \\ &\qquad \geqslant p\left(1 - \frac{q}{2}\right)\left(\frac{1-m_d}{4} + Z^*\right), \end{aligned}$$

for some $Z^* \geqslant 0$, or,

$$Z^* \geqslant \frac{1}{4}\left(m_d \frac{p}{q} - 1\right),$$
$$Z^* \leqslant \frac{q - (1 - m_d)p}{4(p - q)}.$$

When $p = 2q$, both right-hand sides of the above inequalities become $(1/2)(m_d - 1/2)$, but when $p > 2q$, we have that $\frac{1}{4}(m_d\, p/q - 1) > \frac{q-(1-m_d)p}{4(p-q)}$, so the set of values for Z^* that would implement (DE, DE) as an equilibrium is an empty set.

Additionally, for some graphic intuition, we could consider the left panel of figure 1 from the proof of Proposition 2. There, the blue-shaded region is the set of pairs (Z^I, Z^{II}) that support (DE, DE) as an equilibrium. This region lies above the line $Z^I = Z^{II}$, which corresponds to the fixed-prize contest, and strictly above that line when $p > 2q$.

B Online Appendix

B.1 Equilibria without a contest

In a game without a contest, and taking into account optimal pricing in all possible subgames, the expected payoff of firm i which takes an action $a_i \in \{DD, DE, EE\}$, given the action of the opponent j, is given in the following payoff matrix:

		j: DD	DE	EE
i	DD	$\frac{1}{8}p(2-p)m_d$	$\frac{1}{8}p(2-q)m_d$	$p\frac{1}{4}m_d$
	DE	$\frac{1}{8}q(2-m_d \times p)$	$\frac{1}{8}(2-q)q$	$\frac{1}{8}q(2-p(1-m_d))$
	EE	$p\left(\frac{1}{4}(1-m_d)\right)$	$\frac{1}{8}p(2-q)(1-m_d)$	$\frac{1}{8}p(2-p)(1-m_d)$

Table 1: Expected Payoff Matrix without a Contest.

Below, we first provide best replies to various actions of the opponent. Then, we characterize the equilibria, first for the case $p \leq 2q$, then for the case $p > 2q$.

B.1.1 Best replies

Lemma 2. *(DD, DD) is an equilibrium if and only if* $m_d \geq \max\left\{\frac{2q}{p(2-p+q)}, \frac{2}{4-p}\right\}$.

Proof. Suppose that the firm j is playing DD. Then $\Pi_i(DD, DD) \geq \Pi_i(DE, DD)$ if and only if

$$\frac{1}{8}p(2-p)m_d \geq \frac{1}{8}q(2-m_d \times p)$$
$$m_d \geq \frac{2q}{p(2-p+q)}.$$

Similarly, $\Pi_i(DD, DD) \geq \Pi_i(EE, DD)$ if and only if

$$\frac{1}{8}p(2-p)m_d \geq p\left(\frac{1}{4}(1-m_d)\right)$$
$$m_d \geq \frac{2}{4-p}.$$

Thus, (DD, DD) is an equilibrium if and only if

$$m_d \geq \max\left\{\frac{2q}{p(2-p+q)}, \frac{2}{4-p}\right\}.$$

$\square$

Lemma 3. *DE is a best response to DD if and only if* $m_d \in \left[\frac{2(p-q)}{p(2-q)}, \frac{2q}{p(2-p+q)}\right]$. *This interval is empty if* $p > 2q$.

Proof. Suppose that the firm j is playing DD. Analogously to above $\Pi_i(DE, DD) \geq \Pi_i(DD, DD)$ if and only if

$$m_d \leq \frac{2q}{p(2-p+q)}.$$

Similarly, $\Pi_i(DE, DD) \geq \Pi_i(EE, DD)$ if and only if

$$\frac{1}{8}q(2 - m_d \times p) \geq p\left(\frac{1}{4}(1-m_d)\right)$$
$$m_d \geq \frac{2(p-q)}{p(2-q)}.$$

To prove that the interval is empty if $p > 2q$, note that

$$\frac{\partial}{\partial p}\frac{2(p-q)}{p(2-q)} = \frac{2q}{p^2(2-q)} > 0$$
$$\frac{\partial}{\partial p}\frac{2q}{p(2-p+q)} = -\frac{2q(-2p+q+2)}{(-p^2+pq+2p)^2} < 0,$$

and

$$\left.\frac{2(p-q)}{p(2-q)}\right|_{p=2q} = \left.\frac{2q}{p(2-p+q)}\right|_{p=2q}.$$

$\square$

Lemma 4. *EE is a best response to DD if and only if* $m_d \leq \min\left\{\frac{2(p-q)}{p(2-q)}, \frac{2}{4-p}\right\}$.

Proof. Suppose that the firm j is playing DD. Analogously to above $\Pi_i(EE, DD) \geq \Pi_i(DD, DD)$ if and only if

$$m_d \leq \frac{2}{4-p}.$$

Similarly, $\Pi_i(EE, DD) \geq \Pi_i(DE, DD)$ if and only if

$$m_d \leq \frac{2(p-q)}{p(2-q)}.$$

$\square$

Lemma 5. *DD is a best response to DE if and only if* $m_d \geq \frac{q}{p}$. *This condition is always satisfied if* $p > 2q$. *EE is never a best response to DE.*

Proof. Suppose that the firm j is playing DE. Then $\Pi_i(DD, DE) \geq \Pi_i(DE, DE)$ if and only if

$$\begin{aligned}\frac{1}{8}p(2-q)m_d &\geq \frac{1}{8}(2-q)\\ m_d &\geq \frac{q}{p}.\end{aligned}$$

If $p > 2q$, then $\frac{1}{2} > \frac{q}{p}$ and since $m_d > \frac{1}{2}$, then $m_d \geq \frac{q}{p}$ always holds. Next, $\Pi_i(DD, DE) \geq \Pi_i(EE, DE)$ if and only if

$$\begin{aligned}\frac{1}{8}p(2-q)m_d &\geq \frac{1}{8}p(2-q)(1-m_d)\\ m_d &\geq \frac{1}{2},\end{aligned}$$

which holds by assumption. $\square$

Lemma 6. *DD is a best response to EE if and only if $m_d \geq \dfrac{q(2-p)}{p(2-q)}$, which is always satisfied for $p > 2q$. EE is never a best response to EE.*

Proof. Suppose that the firm j is playing EE. Then $\Pi_i(DD, EE) \geq \Pi_i(DE, EE)$ if and only if

$$\begin{aligned}\frac{1}{4}pm_d &\geq \frac{1}{8}q(2-p(1-m_d))\\ 2pm_d &\geq q(2-p+pm_d)\\ m_d(2p-qp) &\geq q(2-p)\\ m_d &\geq \frac{q(2-p)}{p(2-q)}.\end{aligned}$$

To see that this expression is always satisfied for $p > 2q$, notice that $\dfrac{q(2-p)}{p(2-q)}$ is decreasing in p and equals $\dfrac{1-q}{2-q}$ for $p = 2q$, which in turn is below $1/2$ for all admissible values of q. Next, $\Pi_i(DD, EE) \geq \Pi_i(EE, EE)$ if and only if

$$\begin{aligned}\frac{1}{4}pm_d &\geq \frac{1}{8}p(2-p)(1-m_d)\\ 2m_d &\geq 2-p-(2-p)m_d\\ m_d &\geq \frac{2-p}{4-p},\end{aligned}$$

which is always satisfied since the fraction on the right is declining in p and equals $1/2$ for $p = 0$. $\square$

B.1.2 Equilibria when $p \leq 2q$

Lemma 7. *Suppose that $p \leq 2q$ and there is no contest. Then (DD, DD) is an equilibrium if and only if $m_d \geq \dfrac{2q}{p(2-p+q)}$.*

Proof. By Lemma 2, (DD, DD) is an equilibrium if and only if $m_d \geq \max\left\{\dfrac{2q}{p(2-p+q)}, \dfrac{2}{4-p}\right\}$. Since

$$
\begin{aligned}
\frac{2q}{p(2-p+q)} &\geq \frac{2}{4-p} \\
4q - pq &\geq 2p - p^2 + pq \\
4q - 2pq &\geq 2p - p^2 \\
2q(2-p) &\geq p(2-p) \\
2q &\geq p,
\end{aligned}
$$

given the assumption in the lemma $\dfrac{2q}{p(2-p+q)} = \max\left\{\dfrac{2q}{p(2-p+q)}, \dfrac{2}{4-p}\right\}$. □

Lemma 8. *Suppose that $p \leq 2q$ and there is no contest. Then (DD, DE) is an equilibrium if and only if $m_d \in \left[\max\left\{\dfrac{q}{p}, \dfrac{2(p-q)}{p(2-q)}\right\}, \dfrac{2q}{p(2-p+q)}\right]$.*

Proof. By Lemma 3, DE is a best response to DD if $m_d \in \left[\dfrac{2(p-q)}{p(2-q)}, \dfrac{2q}{p(2-p+q)}\right]$. By Lemma 5, DD is a best response to DE if $m_d \geq q/p$. □

Lemma 9. *Suppose that $p \leq 2q$ and there is no contest. Then (DD, EE) is an equilibrium if and only if $m_d \in \left[\dfrac{q(2-p)}{p(2-q)}, \dfrac{2(p-q)}{p(2-q)}\right]$.*

Proof. By Lemma 4, EE is a best response to DD if $m_d \leq \min\left\{\dfrac{2(p-q)}{p(2-q)}, \dfrac{2}{4-p}\right\}$. Since

$$
\begin{aligned}
\frac{2(p-q)}{p(2-q)} &\leq \frac{2}{4-p} \\
(p-q)(4-p) &\leq p(2-q) \\
4p - 4q - p^2 + pq &\leq 2p - pq \\
2p - 4q - p^2 + 2pq &\leq 0 \\
p(2-p) - 2q(2-p) &\leq 0 \\
(2-p)(p-2q) &\leq 0,
\end{aligned}
$$

which always holds as $p \leq 2q$, then EE is a best response to DD if $m_d \leq \dfrac{2(p-q)}{p(2-q)}$. By Lemma 6, DD is a best response to EE if $m_d \geq \dfrac{q(2-p)}{p(2-q)}$. This interval is not empty for $p \leq 2q$, for example, when $q = 5/13$ and $p = 3/4$. □

Lemma 10. *(DE, DE) is an equilibrium if and only if $p \leq 2q$ and $m_d \in (\frac{1}{2}, \frac{q}{p}]$.*

Proof. Analogously to the above $\Pi_i(DE, DE) \geq \Pi_i(DD, DE)$ if and only if $m_d \leq \frac{q}{p}$, which can only be satisfied if $p \leq 2q$. Next, $\Pi_i(DE, DE) \geq \Pi_i(EE, DE)$ if and only if

$$\begin{aligned}\frac{1}{8}(2-q)q &\geq \frac{1}{8}p(2-q)(1-m_d) \\ q &\geq p(1-m_d) \\ m_d &\geq 1 - \frac{q}{p}.\end{aligned}$$

If $p \leq 2q$, then $\frac{1}{2} \leq \frac{q}{p}$, so that $1 - \frac{q}{p} \leq \frac{1}{2}$, which implies that $m_d \geq 1 - \dfrac{q}{p}$ is always satisfied. $\square$

Lemma 11. *Suppose that $p \leq 2q$ and there is no contest. Then neither (DE, EE) nor (EE, EE) are an equilibrium.*

Proof. By Lemma 5 EE is never a best response to DE. By Lemma 6 EE is never a best response to EE. $\square$

B.1.3 Equilibria when $p > 2q$

Lemma 12. *Suppose that $p > 2q$ and there is no contest. Then (DD, DD) is an equilibrium if and only if $m_d \geq \dfrac{2}{4-p}$.*

Proof. By Lemma 2, (DD, DD) is an equilibrium if and only if $m_d \geq \max\left\{\dfrac{2q}{p(2-p+q)}, \dfrac{2}{4-p}\right\}$. Since

$$\begin{aligned}\frac{2q}{p(2-p+q)} &< \frac{2}{4-p} \\ 2q &< p\end{aligned}$$

given the condition $p > 2q$ in the Lemma, then $\dfrac{2}{4-p} = \max\left\{\dfrac{2q}{p(2-p+q)}, \dfrac{2}{4-p}\right\}$. $\square$

Lemma 13. *Suppose that $p > 2q$ and there is no contest. Then (DD, EE) is an equilibrium if and only if $m_d \in \left(\dfrac{1}{2}, \dfrac{2}{4-p}\right]$.*

Proof. By Lemma 4, EE is a best response to DD if $m_d \leq \min\left\{\dfrac{2(p-q)}{p(2-q)}, \dfrac{2}{4-p}\right\}$. Analogously to the proof of Lemma 9, we conclude that when $p > 2q$, EE is a best response to DD if $m_d \leq \dfrac{2}{4-p}$. By Lemma 6, DD is always a best response to EE when $p > 2q$. $\square$

Lemma 14. *Suppose that $p > 2q$ and there is no contest. Then (DE, DD), (DE, DE), (DE, EE) and (EE, EE) are never an equilibrium.*

Proof. By Lemma 3, DE is never a best response to DD. By Lemma 10, (DE, DE) is never an equilibrium if $p > 2q$. By Lemma 5 EE is never a best response to DE. By Lemma 6 EE is never a best response to EE. $\square$

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