Optimal Contest Design: A General Approach

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Introduction

- Many economic interactions can be described as contests:
  - promotions;
  - elections;
  - university entrance exams;
  - innovation competitions;
  - sporting events.
- All of these contests are designed.
- How then should contests be optimally designed?

Introduction

- The usual approach:
  - pick a contest family (e.g. Tullock, Lazear-Rosen, All-Pay),
  - then optimize (usually over prize vectors).
- Intuition we get often does not transfer across families:
  - Tullock $\longrightarrow$ winner-take-all is optimal (Clark and Riis, 1998; Schweinzer and Segev, 2012);
  - All-Pay $\longrightarrow$ $n-1$ equal prizes are optimal (Fang, Noe and Strack, 2020).
- Should we choose Tullock, All-Pay, or some other contest?

This paper

- Provides a general framework where the designer can choose
  - any prize profile and
  - any prize allocation rule (i.e., contest success function),

  including all standard contests as special cases.
- Focuses on the maximization of total effort net of prizes.
- Shows that a nested Tullock contest with $n-1$ equal prizes implements the optimum.
- Considers extensions to imperfect performance measurement and heterogeneous agents.

Model

Environment

- A principal organizes a contest among $n \geq 2$ agents.
- In a contest, each agent
  - chooses an effort $e_i \geq 0$, and
  - obtains a monetary transfer $t_i \geq 0$.
- The payoff of agent i is

$$\pi_i(e_i, t_i) = u(t_i) - c(e_i).$$

  - $u' > 0$, $u'' \leq 0$, and $u(0) = 0$;
  - $c' > 0$, $c'' > 0$, $c(0) = c'(0) = 0$, and $\lim_{e \to \infty} c'(e) = \infty$.

Environment

- Let $e = (e_1, \ldots, e_n)$, $E = \mathbb{R}^n_+$, and $t = (t_1, \ldots, t_n)$.
- The payoff of the principal is

$$\pi_P(e, t) = \sum_{i=1}^{n} e_i - \sum_{i=1}^{n} t_i.$$

- Our results continue to hold with any production function $g : E \to \mathbb{R}_+$ that is symmetric, increasing and concave.

Contests

- A contest (y, μ) is defined by
  - a prize profile $y = (y_1, \ldots, y_n)$, w.l.o.g. $y_1 \geq \ldots \geq y_n$, and
  - a contest success function (CSF) $\mu : E \to \Delta T(y)$,

  where $T(y)$ is the set of all permutations of y.
- A CSF maps every $e \in E$ into a (possibly non-deterministic) allocation of prizes among agents, summarized by μ^e.

Contests

- **Examples.** If $y = (y_1, y_2)$, then $T(y) = \{(y_1, y_2), (y_2, y_1)\}$.

- In an all-pay contest:

$$\mu^e((y_1, y_2)) = \begin{cases} 1 & \text{if } e_1 > e_2, \\ 1/2 & \text{if } e_1 = e_2, \\ 0 & \text{if } e_1 < e_2. \end{cases}$$

- In a Tullock contest with impact function f:

$$\mu^e((y_1, y_2)) = \begin{cases} \frac{f(e_1)}{f(e_1)+f(e_2)} & \text{if } \max\{e_1, e_2\} > 0, \\ 1/2 & \text{otherwise.} \end{cases}$$

Objectives: agents

- Given a contest (y, μ), the agents simultaneously choose their (possibly random) efforts $\sigma_i \in \Delta\mathbb{R}_+$.

- A contest implements a strategy profile $\sigma = (\sigma_1, \ldots, \sigma_n)$ if

$$\mathbb{E}_{\sigma_i, \sigma_{-i}}\left[\mathbb{E}_{\mu^e}\left[\pi_i(e_i, t_i)\right]\right] \geq \mathbb{E}_{\tilde{\sigma}_i, \sigma_{-i}}\left[\mathbb{E}_{\mu^e}\left[\pi_i(e_i, t_i)\right]\right] \quad \forall \tilde{\sigma}_i \;\&\; i.$$

- Since each agent can obtain a payoff of at least zero by choosing zero effort, participation constraints can be ignored.

Objectives: principal

- The principal chooses a contest (y, μ) which implements a strategy profile σ in order to maximize expected total effort net of total transfers.

- Formally, the principal solves

$$\max_{\sigma,(y,\mu)} \mathbb{E}_\sigma \left[\sum_{i=1}^{n} e_i \right] - \sum_{i=1}^{n} y_i$$

  such that (y, μ) implements σ.

- A contest (y^*, μ^*) is optimal if there exists σ^* such that $(\sigma^*, (y^*, \mu^*))$ solves the above problem.

Optimal Contest

Nested Tullock contests

- Nested Tullock was introduced by Clark and Riis (1996).
- With n agents and a single positive prize, the probability that i wins the prize is:

$$p_i(e) = \frac{f(e_i)}{\sum_{j=1}^{n} f(e_j)}. \tag{1}$$

- With multiple positive prizes, (1) is applied in a nested fashion by eliminating the winners in each round sequentially.

Optimal contest

Proposition 1

The following contest (y^*, μ^*) is optimal:

(i) The prize profile is

$$y^* = \left(\frac{x^*}{n-1}, \dots, \frac{x^*}{n-1}, 0\right),$$

where x^* is given by $u'\left(\frac{x^*}{n-1}\right) = c'\left(c^{-1}\left(\frac{n-1}{n} u\left(\frac{x^*}{n-1}\right)\right)\right)$.

(ii) The CSF μ^* is a nested Tullock with the impact function

$$f(e_i) = c(e_i)^{r^*(n)} \quad \text{and} \quad r^*(n) = \frac{n-1}{H_n - 1},$$

where $H_n = \sum_{k=1}^{n} 1/k$ is the n-th harmonic number.

Optimal contest: effort and competitiveness

- The optimal effort profile is symmetric and in pure strategies:

$$c(e^*) = \frac{n-1}{n} u\left(\frac{x^*}{n-1}\right).$$

- In general, $e^* < e^{FB}$ if agents are risk-averse, but efficiency loss vanishes as $n \to +\infty$.

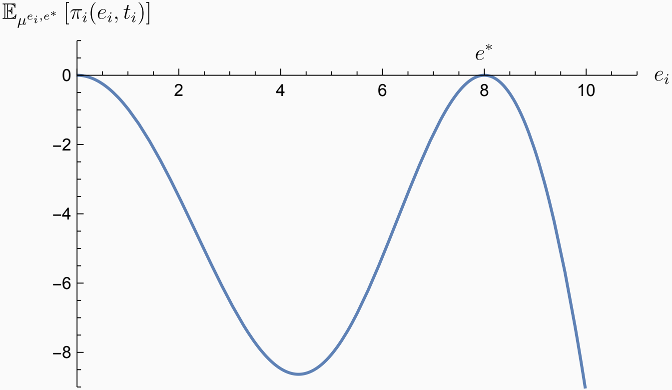
- The precision of the CSF, r^*, measures competitiveness:
  - $r^*(2) = 2$, $r^*(n) \uparrow$ in n, and $\lim_{n\to\infty} r^*(n) = \infty$
  - $r^*(n)$ is such that any increase in the competitiveness of the contest would destroy the pure strategy equilibrium.

Optimal contest: results from the literature

- Take a winner-take-all all-pay contest.
- Fang, Noe and Strack (2020) show that "turning down the heat" by dividing the prize increases the total expected effort.
- They conclude that the optimal all-pay contest has $n - 1$ equal prizes.
- We show that it is beneficial to turn down the heat even further by making the CSF less precise.
- Schweinzer and Segev (2012) show that turning up the heat (by making the prize profile more top-heavy) is beneficial as long as a pure strategy equilibrium exists.
- The optimal "competitiveness" of the contest is exactly at the point where the pure strategy equilibrium appears.

Optimal contest: sketch of the proof

- From Letina, Liu, Netzer (2020) we know that any contest which implements e^* and has a prize vector y^* is optimal.
- Here we show that a nested Tullock CSF is optimal.



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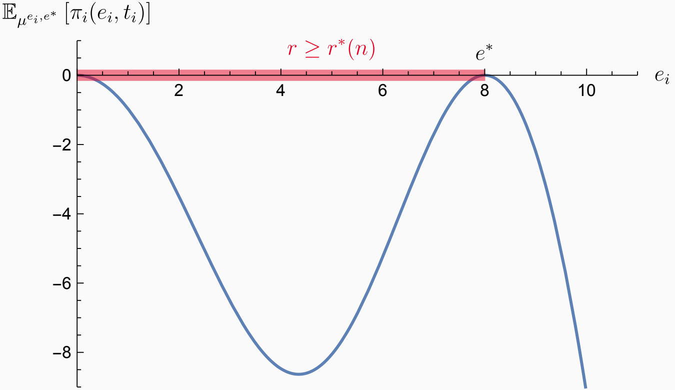

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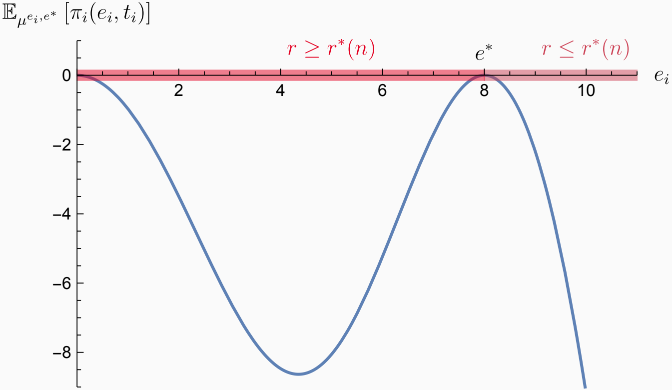

Extensions

Extensions: Imperfect performance measurement

- In the baseline model, effort can be perfectly measured.
- Suppose now effort e can only be imperfectly measured by a signal $s \in S$, which is drawn according to $\eta^e \in \Delta S$.
- We call (S, η) an observational structure.
- What can we say without imposing a specific observational constraint?

Imperfect performance measurement

Proposition 2

Fix an arbitrary observational structure (S, η). A contest with prize profile $y = (x^*/(n-1), \ldots, x^*/(n-1), 0)$ is optimal if it implements $(e^*, \ldots, e^*)$.

- Two examples are provided in the paper:
    - $s_i = e_i \cdot \epsilon_i$, where ϵ_i is drawn from a log-normal distribution;
    - $s = e_1 - e_2$.

Heterogeneous contestants, $n = 2$

Proposition 3

Suppose $n = 2$. For any profile of cost functions (c_1, c_2), the following contest (y^*, μ^*) is optimal:

(i) The prize profile is $y^* = (x^*, 0)$, where x^* is given by

$$(x^*, e_1^*, e_2^*) \in \operatorname*{argmax}_{x, e_1, e_2 \geq 0} \; e_1 + e_2 - x$$
$$\text{s.t.} \quad c_1(e_1) + c_2(e_2) = u(x).$$

(ii) The CSF μ^* is Tullock with agent-specific impact functions

$$f_i(e_i) = \frac{c_i(e_i)^{r_i^*}}{c_i(e_i^*)^{r_i^* - 1}} \quad \text{and} \quad r_i^* = 1 + \frac{c_i(e_i^*)}{c_j(e_j^*)}, \;\; \forall i = 1, 2, j \neq i.$$

Heterogeneous contestants, $n > 2$

Proposition 4

Let $(c_1^m, \ldots, c_n^m) \to (c, \ldots, c)$ uniformly. Then, there exists $\underline{m} \in \mathbb{N}$ such that for all $m \geq \underline{m}$, a contest with $n - 1$ equal positive prizes and one zero prize is optimal.

Concluding remarks

Related literature

- Optimality of contests: Letina, Liu and Netzer (2020).
- Optimal contest in a given class:
  - Tullock: Clark and Riis (1998), Schweinzer and Segev (2012), Fu, Jiao and Lu (2015), Feng and Lu (2018);
  - Lazear-Rosen: Drugov and Ryvkin (2020a), Drugov and Ryvkin (2020b), Morgan, Tumlinson and Vardy (2019);
  - All-pay: Fang, Noe and Strack (2020), Xiao (2016), Olszewski and Siegel (2019), Olszewski and Siegel (2020), Moldovanu and Sela (2001).

Concluding remarks

- We provide a framework that enables us to study contest design, without being restricted to a single class of contests.
- We show that the optimum can be achieved by an appropriately designed Tullock contest.
- The optimal prize vector features a single zero prize and $n - 1$ equal positive prizes.

Concluding remarks

- We focus on optimal design for effort maximization, but there are other objectives that the principal may have.
- For example, in innovation contests the principal is usually only interested in the highest effort.
- In selection contests, where agents' abilities are privately known, the principal may aim to identify the best agent.
- For future research, one can extend our framework to study the design of all these contests.